

REGULAR SEQUENCES AND LOCAL COHOMOLOGY MODULES WITH RESPECT TO A PAIR OF IDEALS

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ABSTRACT. Let R be a Noetherian ring, I and J two ideals of R and t an integer. Let S be the class of Artinian R -modules, or the class of all R -modules N with $\dim_R N \leq k$, where k is an integer. It is proved that $\inf\{i : H_{I,J}^i(M) \notin S\} = \inf\{S - \text{depth}_{\mathfrak{a}}(M) : \mathfrak{a} \in \widetilde{W}(I, J)\}$, where M is a finitely generated R -module, or is a ZD -module such that $M/\mathfrak{a}M \notin S$ for all $\mathfrak{a} \in \widetilde{W}(I, J)$. Let $\text{Supp}_R H_{I,J}^i(M)$ be a finite subset of $\text{Max}(R)$ for all $i < t$. It is shown that there are maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k$ of R such that $H_{I,J}^i(M) \cong H_{\mathfrak{m}_1}^i(M) \oplus H_{\mathfrak{m}_2}^i(M) \oplus \dots \oplus H_{\mathfrak{m}_k}^i(M)$ for all $i < t$.

1. INTRODUCTION

Throughout this paper, R is a commutative Noetherian ring with non-zero identity, I and J are two ideals of R , M is an R -module and s and t are two integers. For notations and terminologies not given in this paper, the reader is referred to [4], [5] and [16] if necessary.

The theory of local cohomology, which was introduced by Grothendieck [10], is a useful tool for attacking problems in commutative algebra and algebraic geometry. Bijan-Zadeh [3] introduced the local cohomology modules with respect to a system of ideals, which is a generalization of ordinary local cohomology modules. As a special case of these extend modules, Takahashi, Yoshino and Yoshizawa [16] defined the local cohomology modules with respect to a pair of ideals. To be more precise, let $W(I, J) = \{\mathfrak{p} \in \text{Spec}(R) : I^t \subseteq J + \mathfrak{p} \text{ for some positive integer } t\}$. The set of elements x of M such that $\text{Supp}_R Rx \subseteq W(I, J)$, is said to be (I, J) -torsion submodule of M and is denoted by $\Gamma_{I,J}(M)$. $\Gamma_{I,J}(-)$ is a covariant, R -linear functor from the category of R -modules to itself. For an integer i , the local cohomology functor $H_{I,J}^i(-)$ with respect to (I, J) , is defined to be the i -th right derived functor of $\Gamma_{I,J}(-)$. Also $H_{I,J}^i(M)$ is called the i -th local cohomology module of M with respect to (I, J) . If $J = 0$, then $H_{I,J}^i(-)$ coincides with the ordinary local cohomology

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functor $H_I^i(-)$. Let $\tilde{W}(I, J) = \{\mathfrak{a} \trianglelefteq R : I^t \subseteq J + \mathfrak{a} \text{ for some positive integer } t\}$. It is easy to see that

$$\Gamma_{I,J}(M) = \{x \in M : \exists \mathfrak{a} \in \tilde{W}(I, J), \mathfrak{a}x = 0\} = \bigcup_{\mathfrak{a} \in \tilde{W}(I, J)} (0 :_M \mathfrak{a}).$$

In section 2, we study extension functors of local cohomology modules with respect to a pair of ideals. Let S be a Melkersson subcategory with respect to I , and M a finitely generated R -module. The current authors, in [13, Theorem 2.11], showed that if $H_{I,J}^i(M) \in S$ for all $i < t$, then $H_I^i(M) \in S$ for all $i < t$. In 2.5, we improve this result and we show that if $\text{Ext}_R^j(N, H_{I,J}^i(M)) \in S$ for all $i < t$ and all $j < t - i$, then $H_{\mathfrak{a}}^i(M) \in S$ for all $i < t$, where M is an arbitrary R -module and N is a finitely generated R -module with $\text{Supp}_R N = V(\mathfrak{a})$ for some $\mathfrak{a} \in \tilde{W}(I, J)$.

Let S be a Serre subcategory of the category of R -modules. Aghapournahr and Melkersson [1] introduced the notion of S -sequences on M as a generalization of regular sequences. Suppose that S is a Melkersson subcategory with respect to I , M is a ZD -module and $M/IM \notin S$. In [14, Theorem 2.9] it is proved that all maximal S -sequences on M in I , have the same length. If this common length is denoted by $S - \text{depth}_I(M)$, then $S - \text{depth}_I(M) = \inf\{i : H_I^i(M) \notin S\}$; see [14, Corollary 2.12]. In 2.25, we generalize this result as follows. Let S be the class of Artinian R -modules, or the class of all R -modules N with $\dim_R N \leq k$, where k is an integer. Then $\inf\{S - \text{depth}_{\mathfrak{a}}(M) : \mathfrak{a} \in \tilde{W}(I, J)\} = \inf\{i : H_{I,J}^i(M) \notin S\}$, where M is a finitely generated R -module, or is a ZD -module such that $M/\mathfrak{a}M \notin S$ for all $\mathfrak{a} \in \tilde{W}(I, J)$.

In section 3, we get some identities between local cohomology modules. Let $\text{Supp}_R H_{I,J}^i(M)$ be a finite subset of $\text{Max}(R)$ for all $i < t$. Then there are maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k$ of R such that $H_{I,J}^i(M) \cong H_{\mathfrak{m}_1}^i(M) \oplus H_{\mathfrak{m}_2}^i(M) \oplus \dots \oplus H_{\mathfrak{m}_k}^i(M)$ for all $i < t$; see 3.7. As a consequence we conclude that, if (R, \mathfrak{m}) is a local ring, then $\inf\{i : H_{I,J}^i(M) \text{ is not Artinian}\} = \inf\{i : H_{I,J}^i(M) \not\cong H_{\mathfrak{m}}^i(M)\}$; see 3.10.

2. EXTENSION FUNCTORS OF LOCAL COHOMOLOGY MODULES

Recall that R is a Noetherian ring, I and J are ideals of R and M is an R -module.

Definition 2.1. A full subcategory of the category of R -modules is said to be Serre subcategory, if it is closed under taking submodules, quotients and extensions. A Serre subcategory S is said to be a Melkersson subcategory with respect to I , if for

any I -torsion R -module M , $0 :_M I \in S$ implies that $M \in S$. A Serre subcategory is called Melkersson subcategory when it is a Melkersson subcategory with respect to all ideals of R .

The class of finitely generated modules and the class of weakly Laskerian modules are Serre subcategories. Aghapournahr and Melkersson [1, Lemma 2.2] proved that if a Serre subcategory is closed under taking injective hulls, then it is a Melkersson subcategory. The class of zero modules, Artinian R -modules, modules with finite support and the class of R -modules N with $\dim_R N \leq k$, where k is a non-negative integer, are Serre subcategories closed under taking injective hulls, and hence are Melkersson subcategories; see [1, Example 2.4]. The class of I -cofinite Artinian modules is a Melkersson subcategory with respect to I , but is not closed under taking injective hulls; see [1, Example 2.5].

The following result is a generalization of [2, Theorem 2.1].

Theorem 2.2. *Let N be an (I, J) -torsion R -module. If $\text{Ext}_R^{t-i}(N, H_{I,J}^i(M)) \in S$ for all $i \leq t$, then $\text{Ext}_R^t(N, M) \in S$.*

Proof. Let $F(-) = \text{Hom}_R(N, -)$ and $G(-) = \Gamma_{I,J}(-)$. Then we have $FG(M) = \text{Hom}_R(N, M)$. By [15, Theorem 11.38], there is the Grothendieck spectral sequence

$$E_2^{p,q} := \text{Ext}_R^p(N, H_{I,J}^q(M)) \Rightarrow \text{Ext}_R^{p+q}(N, M).$$

There is a finite filtration

$$0 = \varphi^{t+1}H^t \subseteq \varphi^t H^t \subseteq \cdots \subseteq \varphi^1 H^t \subseteq \varphi^0 H^t = \text{Ext}_R^t(N, M)$$

such that $E_\infty^{t-i,i} \cong \varphi^{t-i}H^t / \varphi^{t+1-i}H^t$ for all $i \leq t$. It is enough to show that $\varphi^0 H^t \in S$. By hypothesis, $E_j^{t-i,i} \in S$ for all $j \geq 2$ and $i \leq t$, and so $E_\infty^{t-i,i} \in S$ for all $i \leq t$. The sequence

$$0 \longrightarrow \varphi^{t+1-i}H^t \longrightarrow \varphi^{t-i}H^t \longrightarrow E_\infty^{t-i,i} \longrightarrow 0$$

is exact for all $i \leq t$. Therefore it follows that $\varphi^0 H^t \in S$. \square

Corollary 2.3. *Let N be an (I, J) -torsion R -module. If $\text{Ext}_R^j(N, H_{I,J}^i(M)) \in S$ for all $i < t$ and all $j < t - i$, then $\text{Ext}_R^i(N, M) \in S$ for all $i < t$.*

Corollary 2.4. *Suppose that S is a Melkersson subcategory with respect to I , and N is a finitely generated R -module with $\text{Supp}_R N = V(\mathfrak{a})$ for some $\mathfrak{a} \in \widetilde{W}(I, J)$. If*

$\text{Ext}_R^j(N, H_{I,J}^i(M)) \in S$ for all $i < t$ and all $j < t - i$, then $H_{\mathfrak{a}}^i(L, M) \in S$ for all $i < t$ and all finitely generated R -modules L .

Proof. The result follows by 2.3 and [1, Theorem 2.9]. \square

The following result improves [13, Theorem 2.11].

Corollary 2.5. *Suppose that S is a Melkersson subcategory with respect to I , and N is a finitely generated R -module with $\text{Supp}_R N = V(\mathfrak{a})$ for some $\mathfrak{a} \in \widetilde{W}(I, J)$. If $\text{Ext}_R^j(N, H_{I,J}^i(M)) \in S$ for all $i < t$ and all $j < t - i$, then $H_{\mathfrak{a}}^i(M) \in S$ for all $i < t$.*

Proof. In 2.4, put $L = R$. \square

Corollary 2.6. *If S is a Melkersson subcategory with respect to I , then*

$$\inf\{i : H_{I,J}^i(M) \notin S\} \leq \inf\{\inf\{i : H_{\mathfrak{a}}^i(M) \notin S\} : \mathfrak{a} \in \widetilde{W}(I, J)\}.$$

As a generalization of finitely generated modules, Evans [9] introduced ZD -modules as follows.

Definition 2.7. An R -module M is said to be zero-divisor module (ZD -module), if for any submodule N of M , the set $Z_R(M/N)$ is a finite union of prime ideals in $\text{Ass}_R M/N$.

According to [7, Example 2.2], the class of ZD -modules contains finitely generated, Laskerian, weakly Laskerian, linearly compact and Matlis reflexive modules. Also it contains modules whose quotients have finite Goldie dimension and modules with finite support, in particular Artinian modules. Therefore the class of ZD -modules is much larger than that of finitely generated modules.

Definition 2.8. An element a of R is called S -regular on M , if $0 :_M a \in S$. A sequence a_1, \dots, a_t is an S -sequence on M , if a_i is S -regular on $M/(a_1, \dots, a_{i-1})M$ for $i = 1, \dots, t$. The S -sequence a_1, \dots, a_t is said to be maximal S -sequence on M , if a_1, \dots, a_t, y is not an S -sequence on M for any $y \in R$.

When S is the class of zero modules, Artinian R -modules, modules with finite support, and the class of R -modules N with $\dim_R N \leq k$, where k is a non-negative integer, then S -sequences on M are, poor M -sequences, filter-regular sequences,

generalized regular sequences, and M -sequences in dimension $> k$, respectively; see [1, Example 2.8].

Let S be a Melkersson subcategory with respect to I , and M a ZD -module such that $M/IM \notin S$. The current authors, in [14, Theorem 2.9], proved that all maximal S -sequences on M in I , have the same length.

Definition 2.9. Let S be a Melkersson subcategory with respect to I , and M a ZD -module such that $M/IM \notin S$. The common length of all maximal S -sequences on M in I , is denoted by $S - \text{depth}_I(M)$. If $M/IM \in S$, we set $S - \text{depth}_I(M) = \infty$.

Suppose that M is a ZD -module. When S is the class of zero modules, Artinian R -modules, and modules with finite support, then $S - \text{depth}_I(M)$ is the same as ordinary $\text{depth}_I(M)$, $f - \text{depth}_I(M)$ (filter-depth), and $g - \text{depth}_I(M)$ (generalized depth), respectively.

Corollary 2.10. Let S be a Melkersson subcategory with respect to I , and M a ZD -module. Then

$$\inf\{i : H_{I,J}^i(M) \notin S\} \leq \inf\{S - \text{depth}_{\mathfrak{a}}(M) : \mathfrak{a} \in \widetilde{W}(I, J)\}.$$

Proof. The result follows by 2.6 and [14, Corollary 2.12]. \square

In the following, we study the relation between generalized local cohomology modules and local cohomology modules with respect to a pair of ideals.

Corollary 2.11. Suppose that N is a finitely generated \mathfrak{a} -torsion R -module for some $\mathfrak{a} \in \widetilde{W}(I, J)$. If $\text{Ext}_R^{t-i}(N, H_{I,J}^i(M)) \in S$ for all $i \leq t$, then $H_{\mathfrak{a}}^t(N, M) \in S$.

Proof. The result follows by 2.2. Note that $\Gamma_{\mathfrak{a}}(N) \subseteq \Gamma_{I,J}(N)$ and by [8, Lemma 2.1] we have $\text{Ext}_R^i(N, M) \cong H_{\mathfrak{a}}^i(N, M)$ for any integer i . \square

Corollary 2.12. Suppose that N is a finitely generated \mathfrak{a} -torsion R -module for some $\mathfrak{a} \in \widetilde{W}(I, J)$. If $\text{Ext}_R^j(N, H_{I,J}^i(M)) \in S$ for all $i < t$ and all $j < t - i$, then $H_{\mathfrak{a}}^i(N, M) \in S$ for all $i < t$.

The following result is a generalization of [2, Theorem 2.3].

Theorem 2.13. Let N be an (I, J) -torsion R -module. If $\text{Ext}_R^{s+t+1-i}(N, H_{I,J}^i(M)) \in S$ for all $i < t$, $\text{Ext}_R^{s+t-1-i}(N, H_{I,J}^i(M)) \in S$ for all $t < i < s+t$, and $\text{Ext}_R^{s+t}(N, M) \in S$, then $\text{Ext}_R^s(N, H_{I,J}^t(M)) \in S$.

Proof. Let $F(-) = \text{Hom}_R(N, -)$ and $G(-) = \Gamma_{I,J}(-)$. Then we have $FG(M) = \text{Hom}_R(N, M)$. By [15, Theorem 11.38], there is the Grothendieck spectral sequence

$$E_2^{p,q} := \text{Ext}_R^p(N, H_{I,J}^q(M)) \Rightarrow \text{Ext}_R^{p+q}(N, M).$$

There is a finite filtration

$$0 = \varphi^{s+t+1}H^{s+t} \subseteq \varphi^{s+t}H^{s+t} \subseteq \dots \subseteq \varphi^1H^{s+t} \subseteq \varphi^0H^{s+t} = \text{Ext}_R^{s+t}(N, M)$$

such that $E_\infty^{s+t-i,i} \cong \varphi^{s+t-i}H^{s+t} / \varphi^{s+t+1-i}H^{s+t}$ for all $i \leq s+t$. It is enough to show that $E_2^{s,t} \in S$. We have the following exact sequences

$$0 \longrightarrow \text{Ker } d_{t+1-i}^{s,t} \longrightarrow E_{t+1-i}^{s,t} \xrightarrow{d_{t+1-i}^{s,t}} E_{t+1-i}^{s+t+1-i,i}$$

and

$$0 \longrightarrow \text{Im } d_{t+1-i}^{s-t-1+i, 2t-i} \longrightarrow \text{Ker } d_{t+1-i}^{s,t} \longrightarrow E_{t+2-i}^{s,t} \longrightarrow 0$$

for all i . By hypothesis, $E_{t+1-i}^{s+t+1-i,i} \in S$ for all $i < t$, and $E_{1+i-t}^{s+t-1-i,i} \in S$ for all $t < i < s+t$. It follows that $E_{t+1-i}^{s-t-1+i, 2t-i} \in S$ for all $t-s < i < t$. Note that if $i \leq t-s$, then $E_{t+1-i}^{s-t-1+i, 2t-i} = 0$. Hence $E_{t+1-i}^{s-t-1+i, 2t-i} \in S$ for all $i < t$, and therefore $\text{Im } d_{t+1-i}^{s-t-1+i, 2t-i} \in S$ for all $i < t$. Also we have $E_{s+t+2}^{s,t} = E_\infty^{s,t} \in S$, because $E_\infty^{s,t} \cong \varphi^s H^{s+t} / \varphi^{s+1} H^{s+t}$ and $\varphi^s H^{s+t} \subseteq \varphi^0 H^{s+t} = \text{Ext}_R^{s+t}(N, M) \in S$. Now the claim follows by the above exact sequences. \square

Corollary 2.14. *Let N be an (I, J) -torsion R -module. Let $\text{Ext}_R^{j-i}(N, H_{I,J}^i(M)) \in S$ for $j = s+t, s+t+1$ and all $i < t$, and $\text{Ext}_R^{s+t-1-i}(N, H_{I,J}^i(M)) \in S$ for all $t < i < s+t$. Then $\text{Ext}_R^{s+t}(N, M) \in S$ if and only if $\text{Ext}_R^s(N, H_{I,J}^t(M)) \in S$.*

Proof. The claim follows by 2.2 and 2.13. \square

The following result is a generalization of [17, Theorem 2.3].

Corollary 2.15. *Let N be an (I, J) -torsion R -module. If $\text{Ext}_R^{t+1-i}(N, H_{I,J}^i(M)) \in S$ for all $i < t$, and $\text{Ext}_R^t(N, M) \in S$, then $\text{Hom}_R(N, H_{I,J}^t(M)) \in S$.*

Proof. In 2.13, put $s = 0$. \square

Proposition 2.16. *Let $\text{Ext}_R^{t+1-i}(R/\mathfrak{a}, H_{I,J}^i(M))$ be Artinian for all $\mathfrak{a} \in \widetilde{\mathcal{W}}(I, J)$ and all $i < t$, and $\text{Ext}_R^t(R/\mathfrak{a}, M)$ be Artinian for all $\mathfrak{a} \in \widetilde{\mathcal{W}}(I, J)$. Then $\text{Ext}_R^j(R/\mathfrak{a}, H_{I,J}^j(M))$ is Artinian for all $\mathfrak{a} \in \widetilde{\mathcal{W}}(I, J)$ and all $j \in \mathbb{N}_0$.*

Proof. By 2.15, $\text{Hom}_R(R/\mathfrak{a}, H_{I,J}^t(M))$ is Artinian for all $\mathfrak{a} \in \widetilde{W}(I, J)$. Also we know that

$$H_{I,J}^t(M) = \bigcup_{\mathfrak{a} \in \widetilde{W}(I, J)} (0 :_{H_{I,J}^t(M)} \mathfrak{a}) = \bigcup_{\mathfrak{a} \in \widetilde{W}(I, J)} \text{Hom}_R(R/\mathfrak{a}, H_{I,J}^t(M)).$$

Now the claim follows by [11, Theorem 5.1] and [1, Theorem 2.9]. \square

Corollary 2.17. *Let $\text{Ext}_R^i(R/\mathfrak{a}, M)$ be Artinian for all $\mathfrak{a} \in \widetilde{W}(I, J)$ and all $i < t$. Then $\text{Ext}_R^j(R/\mathfrak{a}, H_{I,J}^i(M))$ is Artinian for all $\mathfrak{a} \in \widetilde{W}(I, J)$, all $i < t$ and all $j \in \mathbb{N}_0$.*

Proof. It follows by 2.16 that $\text{Ext}_R^j(R/\mathfrak{a}, \Gamma_{I,J}(M))$ is Artinian for all $\mathfrak{a} \in \widetilde{W}(I, J)$ and all $j \in \mathbb{N}_0$. Now again using of 2.16, implies that $\text{Ext}_R^j(R/\mathfrak{a}, H_{I,J}^1(M))$ is Artinian for all $\mathfrak{a} \in \widetilde{W}(I, J)$ and all $j \in \mathbb{N}_0$. By continuing this process, the claim follows. \square

Lemma 2.18. *If $H_I^{t-i}(H_{I,J}^i(M)) \in S$ for all $i \leq t$, then $H_{I,J}^t(M) \in S$.*

Proof. Let $F(-) = \Gamma_I(-)$ and $G(-) = \Gamma_{I,J}(-)$. Then $FG(M) = \Gamma_{I,J}(-)$. The rest of the proof is similar to that of 2.2. \square

Corollary 2.19. *If $\text{Ext}_R^i(R/\mathfrak{a}, M)$ is Artinian for all $\mathfrak{a} \in \widetilde{W}(I, J)$ and all $i < t$, then $H_{I,J}^i(M)$ is Artinian for all $i < t$.*

Proof. It follows by 2.17 and [1, Theorem 2.9] that $H_I^j(H_{I,J}^i(M))$ is Artinian for all $i < t$ and all $j \in \mathbb{N}_0$. Now the claim follows by 2.18. \square

Proposition 2.20. *Let S be the class of all R -modules N with $\dim_R N \leq k$, where k is an integer. Let $\text{Ext}_R^{t+1-i}(R/\mathfrak{a}, H_{I,J}^i(M)) \in S$ for all $\mathfrak{a} \in \widetilde{W}(I, J)$ and all $i < t$, and $\text{Ext}_R^t(R/\mathfrak{a}, M) \in S$ for all $\mathfrak{a} \in \widetilde{W}(I, J)$. Then $H_{I,J}^t(M) \in S$.*

Proof. By 2.15, $\text{Hom}_R(R/\mathfrak{a}, H_{I,J}^t(M)) \in S$ for all $\mathfrak{a} \in \widetilde{W}(I, J)$. Also we know that

$$H_{I,J}^t(M) = \bigcup_{\mathfrak{a} \in \widetilde{W}(I, J)} (0 :_{H_{I,J}^t(M)} \mathfrak{a}) = \bigcup_{\mathfrak{a} \in \widetilde{W}(I, J)} \text{Hom}_R(R/\mathfrak{a}, H_{I,J}^t(M)).$$

It follows that $H_{I,J}^t(M) \in S$. \square

Corollary 2.21. *Let S be the class of all R -modules N with $\dim_R N \leq k$, where k is an integer. If $\text{Ext}_R^i(R/\mathfrak{a}, M) \in S$ for all $\mathfrak{a} \in \widetilde{W}(I, J)$ and all $i < t$, then $H_{I,J}^i(M) \in S$ for all $i < t$.*

Proof. We know that

$$\Gamma_{I,J}(M) = \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \text{Hom}_R(R/\mathfrak{a}, \Gamma_{I,J}(M)) = \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \text{Hom}_R(R/\mathfrak{a}, M).$$

Therefore $\Gamma_{I,J}(M) \in S$. It follows by 2.20 that $H_{I,J}^1(M) \in S$. By keeping this process, the claim follows. \square

Corollary 2.22. *Let S be the class of Artinian R -modules, or the class of all R -modules N with $\dim_R N \leq k$, where k is an integer. If $H_{\mathfrak{a}}^i(M) \in S$ for all $\mathfrak{a} \in \widetilde{W}(I,J)$ and all $i < t$, then $H_{I,J}^i(M) \in S$ for all $i < t$.*

Proof. The claim follows by 2.19, 2.21 and [1, Theorem 2.9]. \square

Corollary 2.23. *Let S be the class of Artinian R -modules, or the class of all R -modules N with $\dim_R N \leq k$, where k is an integer. Then the following statements are equivalent:*

- (i) $H_{I,J}^i(M) \in S$ for all $i < t$;
- (ii) $H_{\mathfrak{a}}^i(M) \in S$ for all $\mathfrak{a} \in \widetilde{W}(I,J)$ and all $i < t$.

Proof. The claim follows by 2.5 and 2.22. \square

Corollary 2.24. *Let S be the class of Artinian R -modules, or the class of all R -modules N with $\dim_R N \leq k$, where k is an integer. Then*

$$\inf\{i : H_{I,J}^i(M) \notin S\} = \inf\{\inf\{i : H_{\mathfrak{a}}^i(M) \notin S\} : \mathfrak{a} \in \widetilde{W}(I,J)\}.$$

The following result is a generalization of [14, Theorem 2.13].

Theorem 2.25. *Let S be the class of Artinian R -modules, or the class of all R -modules N with $\dim_R N \leq k$, where k is an integer. Let M be a finitely generated R -module, or be a ZD-module such that $M/\mathfrak{a}M \notin S$ for all $\mathfrak{a} \in \widetilde{W}(I,J)$. Then*

$$\inf\{i : H_{I,J}^i(M) \notin S\} = \inf\{S - \text{depth}_{\mathfrak{a}}(M) : \mathfrak{a} \in \widetilde{W}(I,J)\}.$$

Proof. The claim follows by 2.24 and [14, Theorem 2.13]. \square

3. SOME IDENTITIES BETWEEN LOCAL COHOMOLOGY MODULES

Suppose that

$$E^\bullet(M) : 0 \longrightarrow E_R^0(M) \xrightarrow{d^0} E_R^1(M) \longrightarrow \cdots \longrightarrow E_R^i(M) \xrightarrow{d^i} E_R^{i+1}(M) \longrightarrow \cdots (*)$$

is a minimal injective resolution of M , where $E_R^i(M) \cong \bigoplus_{\mathfrak{p} \in \text{Spec}(R)} \mu^i(\mathfrak{p}, M) E_R(R/\mathfrak{p})$ is a decomposition of $E_R^i(M)$ as the direct sum of indecomposable injective R -modules, $E_R(R/\mathfrak{p})$ denotes the injective hull of R/\mathfrak{p} and $\mu^i(\mathfrak{p}, M)$ denotes the i -th Bass number of M with respect to \mathfrak{p} . It follows by [16, Proposition 1.11] that

$$\Gamma_{I,J}(E_R^i(M)) \cong \bigoplus_{\mathfrak{p} \in W(I,J)} \mu^i(\mathfrak{p}, M) E_R(R/\mathfrak{p}).$$

Hence $\text{Supp}_R \Gamma_{I,J}(E_R^i(M)) = \{\mathfrak{p} \in W(I,J) : \mu^i(\mathfrak{p}, M) \neq 0\}$.

The above mentioned results are assumed known through this section.

Theorem 3.1. *Let S be a Serre subcategory closed under taking injective hulls. The following conditions are equivalent:*

- (i) $H_{I,J}^i(M) \in S$ for all $i < t$.
- (ii) $\Gamma_{I,J}(E_R^i(M)) \in S$ for all $i < t$.

Proof. Since $\Gamma_{I,J}(E_R^i(M))$ is injective and $\text{Ker} \Gamma_{I,J}(d^i) = \text{Ker} d^i \cap \Gamma_{I,J}(E_R^i(M))$, thus $\Gamma_{I,J}(E_R^i(M))$ is injective hull of $\text{Ker} \Gamma_{I,J}(d^i)$. Now the claim follows by [12, Lemma 5.4]. We note that the proof of [12, Lemma 5.4] is still valid if the class of Artinian R -modules is replaced by a Serre subcategory that is closed under taking injective hulls. \square

Corollary 3.2. *The following statements are equivalent:*

- (i) $\text{Supp}_R H_{I,J}^i(M)$ is a finite subset of $\text{Max}(R)$ for all $i < t$;
- (ii) $H_{I,J}^i(M)$ is Artinian for all $i < t$.

Corollary 3.3. *If (R, \mathfrak{m}) is a local ring, then*

$$\inf\{i : \text{Supp}_R H_{I,J}^i(M) \not\subseteq \{\mathfrak{m}\}\} = \inf\{i : H_{I,J}^i(M) \text{ is not Artinian}\}.$$

Proposition 3.4. *Let M be a finitely generated R -module, or be a ZD-module such that $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p} \in W(I,J)$. Then*

$$\inf\{i : H_{I,J}^i(M) \neq 0\} = \inf\{\text{depth} M_{\mathfrak{p}} : \mathfrak{p} \in W(I,J)\}.$$

Proof. Let $t = \inf\{\text{depth} M_{\mathfrak{p}} : \mathfrak{p} \in W(I, J)\}$. It follows by [14, Corollary 2.14] that $\mu^i(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in W(I, J)$ and all $i < t$. So $\Gamma_{I, J}(E_R^i(M)) = 0$ for all $i < t$, and hence $H_{I, J}^i(M) = 0$ for all $i < t$. Therefore $t \leq \inf\{i : H_{I, J}^i(M) \neq 0\}$. Now it is enough to show that $H_{I, J}^t(M) \neq 0$. By assumption, there is $\mathfrak{q} \in W(I, J)$ such that $t = \text{depth} M_{\mathfrak{q}}$. It follows by [14, Corollary 2.14] that $\mu^t(\mathfrak{q}, M) \neq 0$. Therefore $\Gamma_{I, J}(E_R^t(M)) \neq 0$, and hence $H_{I, J}^t(M) \neq 0$ by 3.1. \square

We can get a generalization of [6, Theorem 2.4].

Proposition 3.5. *Let M be a finitely generated R -module, or be a ZD-module such that $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p} \in W(I, J) - \text{Max}(R)$. Then*

$$\inf\{i : \text{Supp}_R H_{I, J}^i(M) \not\subseteq \text{Max}(R)\} = \inf\{\text{depth} M_{\mathfrak{p}} : \mathfrak{p} \in W(I, J) - \text{Max}(R)\}.$$

Proof. Let $t = \inf\{i : \text{Supp}_R H_{I, J}^i(M) \not\subseteq \text{Max}(R)\}$. It follows by 3.1 that $\mu^i(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in W(I, J) - \text{Max}(R)$ and all $i < t$, and there is $\mathfrak{q} \in W(I, J) - \text{Max}(R)$ such that $\mu^t(\mathfrak{q}, M) \neq 0$. Now it follows by [14, Corollary 2.14] that $\text{depth} M_{\mathfrak{p}} \geq t$ for all $\mathfrak{p} \in W(I, J) - \text{Max}(R)$, and $\text{depth} M_{\mathfrak{q}} = t$. Therefore $\inf\{\text{depth} M_{\mathfrak{p}} : \mathfrak{p} \in W(I, J) - \text{Max}(R)\} = t$. \square

Theorem 3.6. *Let $\text{Supp}_R H_{I, J}^i(M)$ be a finite subset of $\text{Max}(R)$ for all $i < t$. Then there are maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k \in W(I, J)$ such that $H_{I, J}^i(M) \cong H_{\mathfrak{m}_1 \mathfrak{m}_2 \dots \mathfrak{m}_k}^i(M)$ for all $i < t$.*

Proof. It follows by 3.1 that $\text{Supp}_R \Gamma_{I, J}(E^i(M))$ is a finite subset of $\text{Max}(R)$ for all $i < t$. Let $\text{Supp}_R \Gamma_{I, J}(E^i(M)) = \{\mathfrak{m}_{i1}, \mathfrak{m}_{i2}, \dots, \mathfrak{m}_{ik_i}\}$, where k_i is an integer. Then

$$\Gamma_{I, J}(E^i(M)) \cong \bigoplus_{j=1}^{k_i} \mu^i(\mathfrak{m}_{ij}, M) E_R(R/\mathfrak{m}_{ij})$$

for all $i < t$. Put $\mathfrak{a} = \prod_{i, j} \mathfrak{m}_{ij}$. Then $V(\mathfrak{a}) = \{\mathfrak{m}_{ij} : 0 \leq i < t, 1 \leq j \leq k_i\}$, and $\mathfrak{a} \in \widetilde{W}(I, J)$. Therefore

$$\begin{aligned} \Gamma_{\mathfrak{a}}(E^i(M)) &\cong \bigoplus_{\mathfrak{p} \in V(\mathfrak{a})} \mu^i(\mathfrak{p}, M) E_R(R/\mathfrak{p}) \\ &= \bigoplus_{j=1}^{k_i} \mu^i(\mathfrak{m}_{ij}, M) E_R(R/\mathfrak{m}_{ij}) \cong \Gamma_{I, J}(E^i(M)) \end{aligned}$$

for all $i < t$, and hence $H_{I, J}^i(M) \cong H_{\mathfrak{a}}^i(M)$ for all $i < t$. \square

Corollary 3.7. *Let $\text{Supp}_R H_{I,J}^i(M)$ be a finite subset of $\text{Max}(R)$ for all $i < t$. Then there are maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k \in \mathbb{W}(I, J)$ such that*

$$H_{I,J}^i(M) \cong H_{\mathfrak{m}_1}^i(M) \oplus H_{\mathfrak{m}_2}^i(M) \oplus \dots \oplus H_{\mathfrak{m}_k}^i(M)$$

for all $i < t$.

Proof. The claim follows by 3.6 and the Mayer-Vietoris sequence [4, 3.2.3]. \square

Corollary 3.8. *Let (R, \mathfrak{m}) be a local ring. If $\text{Supp}_R H_{I,J}^i(M) \subseteq \{\mathfrak{m}\}$ for all $i < t$, then $H_{I,J}^i(M) \cong H_{\mathfrak{m}}^i(M)$ for all $i < t$.*

Corollary 3.9. *If (R, \mathfrak{m}) is a local ring, then*

$$\inf\{i : \text{Supp}_R H_{I,J}^i(M) \not\subseteq \{\mathfrak{m}\}\} = \inf\{i : H_{I,J}^i(M) \not\cong H_{\mathfrak{m}}^i(M)\}.$$

The following result is a generalization of [6, Proposition 2.5].

Corollary 3.10. *If (R, \mathfrak{m}) is a local ring, then*

$$\inf\{i : H_{I,J}^i(M) \text{ is not Artinian}\} = \inf\{i : H_{I,J}^i(M) \not\cong H_{\mathfrak{m}}^i(M)\}.$$

Proof. The claim follows by 3.3 and 3.9. \square

Corollary 3.11. *Let (R, \mathfrak{m}) be a local ring. If $\text{Supp}_R H_{I,J}^i(M) \subseteq \{\mathfrak{m}\}$ for all $i < t$, then $H_{I,J}^i(M) \cong H_{\mathfrak{a}}^i(M)$ for all $\mathfrak{a} \in \widetilde{\mathbb{W}}(I, J)$ and all $i < t$.*

Proof. It follows by 2.5 that $\text{Supp}_R H_{\mathfrak{a}}^i(M) \subseteq \{\mathfrak{m}\}$ for all $\mathfrak{a} \in \widetilde{\mathbb{W}}(I, J)$ and all $i < t$. Therefore $H_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{m}}^i(M)$ for all $\mathfrak{a} \in \widetilde{\mathbb{W}}(I, J)$ and all $i < t$, by 3.8. Now again using of 3.8 implies that $H_{I,J}^i(M) \cong H_{\mathfrak{m}}^i(M)$ for all $i < t$, and the claim follows. \square

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